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## Meshfree Methods

Saeid Zahiri  
Shiraz University,  
Islamic Republic Of Iran

### 1. Introduction

In this chapter we will describe about numerical simulation with meshfree methods. We know; phenomena in nature, whether physical, geological, mechanical, electrical, or biological, can often be describe by means of algebraic, differential, or integral equations. Obtaining exact solutions for these equations is ideal. Unfortunately, we can only obtain exact ones for limited practical problems because most of these problems are complex. Therefore using numerical procedure to obtain approximate solutions is inevitable. One of the most important tools in the field of numerical methods that has been developed newly is meshfree or meshless methods.

A *meshfree method* is a method used to establish system algebraic equations for the whole domain of problem without using a predefined mesh for the domain discretization. This infant method uses a set of scattered nodes, called field nodes, to establish the problem domain and boundaries, which do not require any priori information on the relationship between the nodes for the interpolation or approximation of the unknown functions of field variables. In the FEM, a continuum with a complicated shape is divided into elements, *finite elements*. The individual elements are connected together by a topological map called a *mesh*. Meshfree methods have been proposed and achieved remarkable progress over the past few years. According to the formulation procedure, meshfree methods fall into three categories: meshfree weak form methods (like: EFG, MLPG, LRPIM,...), meshfree strong form methods (like: SPH, Collocation method,...) and meshfree weak-strong form methods based on the combination of both weak form and strong form (like: MWS method). These three categories and their limitations, applications, advantages and other descriptions will be introduced. In seeking for an approximate solution to the problem governed by PDEs and boundary conditions, one first needs to approximate the unknown field function using shape (trial or base) functions before any formulation procedure can be applied to establish the discretized system equations. In this chapter definition of base and shape functions and various techniques for meshfree shape function constructions are discussed. These shape functions are locally supported, because only a set of field nodes in a small local domain are used in the construction and the shape function is not used or regarded as zero outside the local domain. Such a local domain is termed the support domain or influence domain. The concept and kinds of support domain and determination of the dimension of the support domain will be described.

After introducing the concept of support domain, the point interpolation method (PIM) in detail will be discussed. Point interpolation method is one of the series representation

methods for the function approximation, and useful for creating meshfree shape functions. A scalar function defined in the problem domain that is represented by a set of scattered nodes will be shown. There are two types of PIM shape functions have been developed so far using different forms of basis functions Polynomial basis functions and radial basis functions (RBF) have often been used in meshfree methods. These two types of PIMs will be discussed in the following chapter.

For satisfying the boundary conditions, penalty method, direct method, lagrange multiplier method and direct interpolation method can be used to enforce essential boundary conditions. One of these methods due to using meshfree method can be elected and they will be explained and compared.

To simulate some problems, the partial differential equations and boundary conditions for two dimensional solid mechanics and fluid mechanics problem and heat transfer problem especially thermodynamics of plates and shells will be given in sub-sections. These problems are solved with meshfree methods.

## 2. Meshfree methods categories

According to the formulation procedure, meshfree methods fall into three categories: meshfree weak form methods (like: EFG, MLPG, LRPIM,...), meshfree strong form methods (like: Collocation method, SPH,...) and meshfree weak-strong form methods based on the combinations of both weak forms and strong forms (like: MWS method).

### 2.1 Strong form methods

Regarding to formulate the governing equations, the direct approximate solution from the differential equations is used. It means the strong form of governing equations for boundary conditions are directly discretized at the field nodes to obtain a set of discretized system equations. If Taylor series is used and the differentiations are replaced, the method is the strong form method. The strong form method does not need the numerical integration. Thus The background mesh even locally is not needed for the strong form methods.

Meshfree strong form methods have some attractive advantages: a simple algorithm, computational efficiency, and truly meshfree. However, Meshfree strong form methods are often unstable, not robust, and inaccurate, especially for problems with derivative boundary conditions. Several strategies may be used to impose the derivative (Neumann) boundary conditions in the strong form methods, such as the use of fictitious nodes, the use of the Hermite-type Meshfree shape functions, the use of a regular grid on the derivative boundary.

### 2.2 Weak form methods

In Meshfree weak form methods, the governing partial differential equations (PDEs) with derivative boundary conditions are first transformed to a set of so called weak form integral equations using different techniques. The weak forms are then used to derive a set of algebraic system equations through a numerical integration process using sets of background cells that may be constructed globally or locally in the problem domain. Meshfree weak form methods were relatively under developed before 1990, but there has been a substantial increase in research effort since then.

There are now many different versions of Meshfree weak form methods. Meshfree weak form methods based on the global weak forms are called Meshfree global weak form methods, and

those based on local weak forms are called Meshfree local weak form methods. Meshfree global weak form methods are based on the global Galerkin weak form for equations of problems and the Meshfree shape functions. Two typical Meshfree global weak form methods: the element free Galerkin (EFG) method (Belytschko et al., 1994a) and the radial point interpolation method (RPIM) (GR Liu and Gu, 2001c; Wang and GR Liu, 2000; 2002a).

Another typical Meshfree global weak form method is the reproducing kernel particle method (RKPM) proposed by Liu and coworkers in 1995 (Liu et al., 1995). The main idea of RKPM is to improve the SPH approximation to satisfy consistency requirements using a correction function. RKPM has been used in nonlinear and large deformation problems (Chen et al., 1996; Chen et al., 1998; Liu and Jun, 1998), inelastic structures (Chen et al., 1997), structural acoustics (Uras et al., 1997), fluid dynamics (Liu and Jun et al., 1997), et cetera. Meshfree local weak form methods were developed by Atluri and coworkers based on the local Petrov-Galerkin weak form, and the Meshfree shape functions. Some other Meshfree weak form methods have also been developed, such as the hp-cloud method (Armando and Oden, 1995), the partition of unity finite element method (PUFEM) (Melenk and Babuska, 1996; Babuska and Melenk, 1997), the finite spheres method (De and Bathe, 2000), the free mesh method (Yagawa and Yamada, 1996), et cetera.

### 2.3 Weak-strong form methods

These Meshfree methods are called Meshfree weak-strong (MWS) form methods in this book because are based on the combination of weak and strong form methods. The MWS method was developed by GR Liu and Gu (2002d, 2003b). The key idea of the MWS method is that in establishing the discretized system equations, both the strong form and the local weak form are used for the same problem, but for different groups of nodes that carries different types of equations/conditions. The local weak form is used for all the nodes that are on or near boundaries with derivative (Neumann) boundary conditions. The strong form is used for all the other nodes. The MWS method uses least background cells for the integration, and it is currently the almost ideal Meshfree method that can provide stable and accurate solutions for mechanics problems.

There are also Meshfree methods based on the integral representation method for function approximations, such as the Smooth Particle Hydrodynamics (SPH) methods (Lucy, 1977; Gingold and Monaghan, 1977; GR Liu and Liu, 2003, etc.). In the standard SPH method, the function approximation is performed in a weak (integral) form, but strong form equations are directly discretized at the particles.

### 2.4 Comparisons between three meshfree categories

Each meshfree method has features with advantages and defects. With these properties, the appropriate method can be selected to solve the problem. The features of methods are presented in sub-sections.

#### 2.4.1 Advantages and disadvantages

Convergence rate and highest accuracy are important properties in numerical methods. When the problems include Dirichlet boundary conditions, the strong form methods are the best but in case of Neumann boundary conditions, weak form methods are optimum and when the both of Dirichlet and Neumann boundaries are used in problems, the weak-strong form methods are useful.

The strong form methods are with good convergence rate and they are truly meshless. The procedure is straightforward, and the algorithms and coding are simple. They are computationally efficient, and the solution is accurate when there are only Dirichlet boundary conditions.

However, Meshfree strong form methods have disadvantages: they are often unstable and less accurate, especially for problems governed by PDEs<sup>1</sup> with derivative boundary conditions. Derivative boundary conditions (DBC) involve a set of separate differential equations defined on the boundary; these are different from the governing equations defined in the problem domain. These DBCs require special treatments. Unlike integration, which is a smoothing operator, differentiation is a roughening operator; it magnifies errors in an approximation. This magnified error is partially responsible for the instability of the solution of PDEs. Hence, Meshfree strong form methods are often unstable. Special treatments are employed to implement the derivative boundary conditions in Meshfree strong form methods. However, such treatments cannot always control the error. A technique suitable for one problem may not work for another, even one of the same types. A set of parameters tuned for one problem may not work for another.

The common feature of Meshfree weak form methods is that the PDE of a problem is first replaced by or converted into an integral equation (global or local) based on a principle (weighted residual methods, energy principle). Weak form system equations can then be derived by integration by parts. A set of system equations of Meshfree weak form methods can be obtained from the discretization of the weak form using meshfree interpolation techniques. There are four features of the local weak form. The integral operation can smear the error over the integral domain and, therefore improve the accuracy in the solution. It acts like some kind of regularization to stabilize the solution. The requirement of the continuity for the trial function is reduced or weakened, due to the order reduction of the differential operation resulting from the integration by parts. The force (derivative) boundary conditions can be naturally implemented using the boundary integral term resulting from the integration by parts. The system equations in the domain and the derivative boundary conditions are conveniently combined into one single equation.

These features give Meshfree weak form methods the following advantages. They exhibit good stability and excellent accuracy for many problems. The derivative (Neumann) boundary conditions can be naturally and conveniently incorporated into the same weak form equation. No additional equations or treatments are needed and no errors are introduced in the enforcement of traction boundary conditions. A method developed properly using a weak form formulation is applicable to many other problems. A set of parameters tuned for one method for a problem can be used for a wide range of problems. This robustness of the weak form methods have been demonstrated through many practical problems. It is this robustness that makes the weak form methods applicable to many practical engineering problems.

However, Meshfree global weak form methods are meshfree only in terms of the interpolation of the field variables. Background cells have to be used to integrate a weak form over the global domain. The numerical integration makes them computationally expensive, and the background mesh for the integration means that the method is not truly meshless.

In the Meshfree local weak form methods, the local integral domain in the interior of the problem domain is usually of a regular shape. It can be as simple as possible and can be

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<sup>1</sup>Partial Differential Problems



automatically constructed in the process of computation. The Meshfree local weak form methods have obtained satisfactory results in solid mechanics and fluid mechanics (Atluri and Shen, 2002; GR Liu, 2002).

Although the Meshfree local weak form methods made a significant step in developing ideal meshfree methods, the numerical integration is still burdensome, especially for nodes on or near boundaries with complex shape. The local integration can still be computationally expensive for some practical problems. It is therefore desirable to minimize the need for numerical integrations.

The Mesh Weak-Strong method is designed to combine the advantages of strong form and weak form methods and to avoid their shortcomings. This can be performed only after a thorough examination of the features of both types of methods, presented in the above sentences. An Meshfree weak-strong (MWS) form method was proposed recently by GR Liu and Gu (2002d); it aimed to remove the background mesh for integration as much as possible, and yet to obtain stable and accurate solutions even for PDEs with derivative boundary conditions. The MWS method has been successfully developed and used in solid mechanics (Gu and GR Liu, 2005; GR Liu and Gu, 2003b) and fluid mechanics (GR Liu and Wu et al., 2004; GR Liu and Gu et al., 2003c).

The convergence of the MWS method is studied numerically by comparison with other methods. The weak form method treats the Neumann boundary condition naturally and easily. In addition, the accuracy achieved by meshfree methods based on the weak form equations are generally much better than those based on strong form equations. However, the efficiency is a big problem for the weak form methods because of the need for weak form integration.

The MWS method proposed by Liu and Gu was based on both collocation and local radial point interpolation formulation. In the present MWS method, the strong form of meshfree collocation method is applied to the internal nodes and the nodes on the essential boundaries, while the local radial point interpolation weak form is applied to the nodes on the natural boundaries. The advantages of this MWS method are:

1. The Neumann boundary condition can be imposed straightforwardly and accurately with arbitrary nodal distributions.
2. Stable and accurate solution can be obtained with high efficiency.

### 2.4.2 Applications of each category

Strong form methods are suitable for Dirichlet boundary conditions problems and weak form methods are used more with problems that have Neumann boundary conditions. Weak-strong form methods are appropriate for problems with both of Dirichlet and Neumann boundary conditions.

## 3. Shape functions

In seeking for an approximate solution to a problem governed by PDEs and boundary conditions, one first needs to approximate the equation variables using shape functions, before any formulation procedure can be applied to establish the discretized system equations.

This section discusses various techniques for MFree shape function constructions. These shape functions are locally supported, because only a set of field nodes in a small local domain are used in the construction and the shape function is not used or regarded as zero

outside the local domain. Such a local domain is termed the support domain or influence domain or smoothing domain.

### 3.1 Point interpolation methods shape functions

The point interpolation method (PIM) is one of the series representation methods for the function approximation, and is useful for creating Meshfree shape functions. Consider a scalar function  $T(x)$  defined in the problem domain  $\Omega$  that is represented by a set of scattered nodes. The PIM approximates  $T(x)$  at a point of interest  $x$  in the form of

$$T(x) = \sum_{i=1}^m B_i(x) a_i \quad (1)$$

where the  $B_i(x)$  are the basis function defined in the space Cartesian coordinates  $X^T = [x, y]$ ,  $m$  is the number of basis functions, and the  $a_i$  are the coefficients.

For function approximation, a local support domain is first formed for the point of interest at  $x$  which includes a total of  $n$  field nodes. For the conventional point interpolation method (PIM),  $n=m$  is used that results in the conventional PIM shape functions that pass through the function values at methods. The RPIM interpolation augmented with each scattered node within the defined support domain.

For the weighted least square (WLS) approximation or the moving least squares (MLS) approximation,  $n$  is always larger than  $m$ . There are two types of PIM shape functions have been developed so far using different forms of basis functions. Polynomial basis functions (GR Liu and Gu, 1999; 2001a) and radial basis functions (RBF) (Wang and GR Liu, 2000; GR Liu, 2002) have often been used in Meshfree methods.

#### 3.1.1 Conventional polynomial PIM

Using polynomials as the basis functions in the interpolation is one of the earliest interpolation schemes. It has been widely used in establishing numerical methods, such as the FEM. Consider a continuous function  $u(x)$  defined in a domain  $\Omega$ , which is represented by a set of field nodes. The  $u(x)$  at a point of interest  $x$  is approximated in the form of

$$u(x) = \sum_{i=1}^m p_i(x) a_i = \{p_1(x) \ p_2(x) \ \dots\} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \end{pmatrix} = P^T a \quad (2)$$

$$P^T(x) = (1 \ x \ y \ x^2 \ xy \ y^2) \quad \text{for } m=6, p=2 \text{ (2-D)} \quad (3)$$

where  $p_i(x)$  is a given monomial in the polynomial basis function in the space coordinates  $x^T = [x, y]$ ,  $m$  is the number of monomials, and  $a_i$  is the coefficient for  $p_i(x)$  which is yet to be determined. The  $p_i(x)$  in Equation is built using Pascal's triangles, and a complete basis is usually (but not always)

#### 3.1.2 Radial point interpolation shape functions

In order to avoid the singularity problem in the polynomial PIM, the radial basis function (RBF) is used to develop the radial point interpolation method (RPIM) shape functions for Meshfree weak form methods (GR Liu and Gu, 2001c; Wang and Liu, 2000; 2002a,c). The RPIM shape functions will be used for both Meshfree weak form and strong form polynomials can be written as

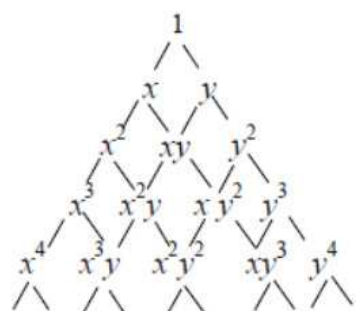


Fig. 1. Pascal-khayyam triangle of monomials for two dimensional domain

$$u(x,y)=\sum_{j=1}^n R_j(x,y)b_j + \sum_{i=1}^m p_i(x,y)a_i = R^T B + P^T A \tag{4}$$

Where  $R_j(x)$  is a radial basis function (RBF),  $n$  is the number of RBFs,  $p_i(x)$  is monomial in the space coordinates  $x^T=[x, y]$ , and  $m$  is the number of polynomial basis functions. When  $m=0$ , pure RBFs are used. Otherwise, the RBF is augmented with  $m$  polynomial basis functions. Coefficients  $a_i$  and  $b_j$  are constants.  $r$  is the distance between the point of interest  $(x,y)$  and a node  $(x_i,y_i)$  at

$$r=\sqrt{(x-x_i)^2+(y-y_i)^2} \tag{5}$$

There are a number of types of radial basis functions (RBF), and the characteristics of RBFs have been widely investigated (Kansa,1990; Sharan et al.,1997; Franke and Schaback, 1997; etc). Four often used RBFs, the multi-quadrics (MQ) function, the Gaussian (Exp) function, the thin plate spline (TPS) function, and the Logarithmic radial basis function, are listed in Table.1.

	Name	†Expression	Shape Parameters
1	Multi-quadrics (MQ)	$R_i(x,y)=(r_i^2+(\alpha_c d_c)^2)^q$	$\alpha_c \geq 0, q$
2	Gaussian (EXP)	$R_i(x,y)=\exp[-\alpha_c(\frac{r_i}{d_c})^2]$	$\alpha_c$
3	Thin Plate Spline (TPS)	$R_i(x,y)=r_i^\eta$	$\eta$
4	Logarithmic	$R_i(x,y)=r_i^\eta \log r_i$	$\eta$

Table 1. Typical radial basis functions with dimensionless shape parameters

Note:  $d_c$  is a characteristic length that relates to the nodal spacing in the local support domain of the point of interest  $x$ , and it is usually the average nodal spacing for all the nodes in the local support domain.

$$R_0 = \begin{bmatrix} R_1(r_1) & R_2(r_1) & \cdots & R_n(r_1) \\ R_1(r_2) & R_2(r_2) & & R_n(r_2) \\ \vdots & & & \vdots \\ R_1(r_n) & R_2(r_n) & & R_n(r_n) \end{bmatrix} \tag{6}$$

the polynomial moment matrix is



$$P_m^T = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \\ P_m(x_1) & P_m(x_2) & & P_m(x_n) \end{bmatrix} \quad (7)$$

$$r_k = \sqrt{(x_k - x_i)^2 + (y_k - y_i)^2} \quad (8)$$

$$\tilde{U}_S = \begin{bmatrix} U_S \\ 0 \end{bmatrix} = \begin{bmatrix} R_0 & P_m \\ P_m^T & 0 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = G a_0 \quad (9)$$

$$u(x) = \begin{Bmatrix} R^T(x) & P^T(x) \end{Bmatrix} G^{-1} \tilde{U}_S = \tilde{\Phi}^T(x) \tilde{U}_S \quad (10)$$

$$\begin{aligned} \tilde{\Phi}^T(x) &= \begin{Bmatrix} R^T(x) & P^T(x) \end{Bmatrix} G^{-1} = \\ &= \{\varphi_1(x) \ \varphi_2(x) \cdots \varphi_n(x) \ \varphi_{n+1}(x) \cdots \varphi_{n+m}(x)\} \end{aligned} \quad (11)$$

The above equations are brought to show the procedure of shape function produce. The shape functions  $\Phi$  are obtained and then the discretized derivatives can be used to governing equations and the parameters are shown with the equation

$$u(x) = \Phi^T(x) U_S = \sum_{i=1}^n \phi_i u_i \quad (12)$$

The derivatives of  $u(x)$  are easily obtained as

$$u_{,l}(x) = \Phi_{,l}^T(x) U_S \quad (13)$$

where  $l$  denotes either the coordinates  $x$  or  $y$ .

### 3.2 Support domain

The accuracy of interpolation for the point of interest depends on the nodes in the support domain as shown in Fig.2. Therefore, a suitable support domain should be chosen to ensure an efficient and accurate approximation. For a point of interest at  $x_q$ , the dimension of the support domain  $d_c$  is determined by

$$\begin{aligned} r_{sx} &= \alpha_s \cdot d_{cx} \\ r_{sy} &= \alpha_s \cdot d_{cy} \end{aligned} \quad (14)$$

$\alpha_s$  is the dimensionless size of the support domain, and  $d_c$  is the nodal spacing near the point at  $x_q$ . If the nodes are uniformly distributed,  $d_c$  is simply the distance between two neighboring nodes. When nodes are non uniform and where  $\alpha$  is a constant of shape parameter,  $dc$  can be defined as an average nodal spacing in the support domain of  $x_q$ . The exponential function of the support domain  $\alpha_s$  controls the actual dimension of the support domain.

Rectangular support domains ( $r_{sx}$  and  $r_{sy}$ : dimensions of the support domain in  $x$  and  $y$  directions). The support domain is centred by  $x_q$ .

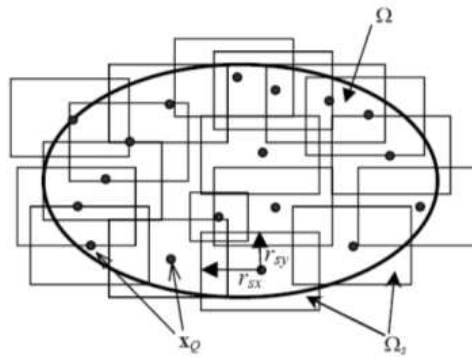


Fig. 2. Support domains of points of interest at  $x_Q$  in Meshfree models.

The actual number of nodes,  $n$ , can be determined by counting all the nodes included in the support domain. Generally, an  $\alpha_s=2.0\sim3.0$  leads to good results for many problems that we have studied. Note that the support domain is usually centered by a point of interest at  $x_Q$ .

#### 4. Satisfying boundary conditions

For the Dirichlet boundary condition, the essential boundary conditions for  $u$  can be simply given as follows: (when node is on the boundary)

$$u=\bar{T} \quad (15)$$

The essential boundary condition can be directly imposed using the direct interpolation method. another method is the Penalty method has been used to enforce essential boundary conditions in the MLPG and LRPIM Methods. Since RPIM shape functions possess the Kronecker delta function property, the essential boundary conditions can be easily enforced as in the FEM (see, e.g., GR Liu and Quek, 2003).

The natural boundary conditions can be satisfy automatically when we use weak-strong form method and no additional equation or treatment is needed.

##### 4.1 Direct method

The  $i$ th component is prescribed by setting

$$u_i=\bar{T}_i \quad (16)$$

Such an essential boundary condition can then be enforced directly into the system Equation through the following modifications to the global matrix and the global right vector. The global matrix,  $K$ , is changed to

$$K = \begin{bmatrix} k_{11} & \dots & k_{1(i-1)} & 0 & k_{1(i+1)} & \dots & k_{1n} \\ \vdots & & & 0 & & & \vdots \\ k_{(i-1)1} & & & 0 & & & k_{(i-1)n} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ k_{(i+1)1} & & & 0 & & & k_{(i+1)n} \\ \vdots & & & 0 & & & \vdots \\ k_{n1} & \dots & k_{n(i-1)} & 0 & k_{n(i+1)} & \dots & k_{nn} \end{bmatrix} \quad (17)$$

The components in the global right vector are changed to

$$F_i = \begin{cases} \bar{T}_i & i = j \\ F_j - k_{ji}\bar{T}_i & i \neq j \end{cases} \quad (18)$$

The direct method can exactly enforce essential boundary conditions, but changing matrices and vectors needs additional computational operations. In addition, the algorithm of the direct method is also complicated.

#### 4.2 Penalty method

The penalty method is a convenient alternative for enforcing the essential boundary conditions, in which the diagonal entry  $k_{ii}$  in the stiffness matrix, is changed to

$$k_{ii} = \alpha.k_{ii} \quad (19)$$

where  $\alpha$  is the penalty coefficient that is the much larger number than the components of the global matrix  $K$ . In the global right vector  $F$ , only the component  $F_i$  is changed as follows

$$F_i = \begin{cases} \alpha.k_{ii}.\bar{T}_i & i = j \\ F_j & i \neq j \end{cases} \quad (20)$$

The penalty method has some advantages: there are only two changes of matrices, and the algorithm is very simple. However, the penalty method can only approximately satisfy the essential boundary conditions. In addition, the accuracy is affected by selection of the penalty coefficient.

the global matrix,  $K$ , is then changed to

$$K = \begin{bmatrix} k_{11} & \dots & k_{1(i-1)} & k_{1i} & k_{1(i+1)} & \dots & k_{1n} \\ \vdots & & & \vdots & & & \vdots \\ k_{(i-1)1} & & & k_{(i-1)i} & & & k_{(i-1)n} \\ k_{i1} & \dots & k_{i(i-1)} & \alpha k_{ii} & k_{i(i+1)} & \dots & k_{in} \\ k_{(i+1)1} & & & k_{(i+1)i} & & & k_{(i+1)n} \\ \vdots & & & \vdots & & & \vdots \\ k_{n1} & \dots & k_{n(i-1)} & k_{ni} & k_{n(i+1)} & \dots & k_{nn} \end{bmatrix} \quad (21)$$

### 5. Examples for numerical simulations

In this section, some problems are brought to show the abilities of meshfree methods for solving the heat, solid and fluid mechanics problems.

#### 5.1 Heat conduction

Meshfree methods are used to solve the heat transfer problem. For example, heat conduction in the plate is solved.

5.1.1 Formulation of heat transfer in the plate

Formulation of heat transfer in Cartesian coordinate is

$$\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})+q'=\rho C_p(\frac{\partial T}{\partial t}) \tag{22}$$

If  $k$ (conductivity coefficient) is constant and for steady state and without any energy generation we have:

$$(\frac{\partial^2 T}{\partial x^2})+(\frac{\partial^2 T}{\partial y^2})=0 \tag{23}$$

$T$  is the Temperature and  $q'$  is the rate of energy generation and  $\rho$  is the density and  $C_p$  is the specific heat in the formula.

5.1.2 Numerical results and discussion Domain representation for heat transfer

First, the temperature distribution in square plate is obtained. In problem 1, the bottom wall is in temperature  $T_0$  and other walls are in temperature 0 and in problem 2, the up wall has Neumann boundary condition. To check the validity of the method, three different problems are considered. Fig.3 shows the domain representation for problems 1 and 2 by the scattered nodes. The essential and natural boundary conditions should be satisfied on the boundary nodes.

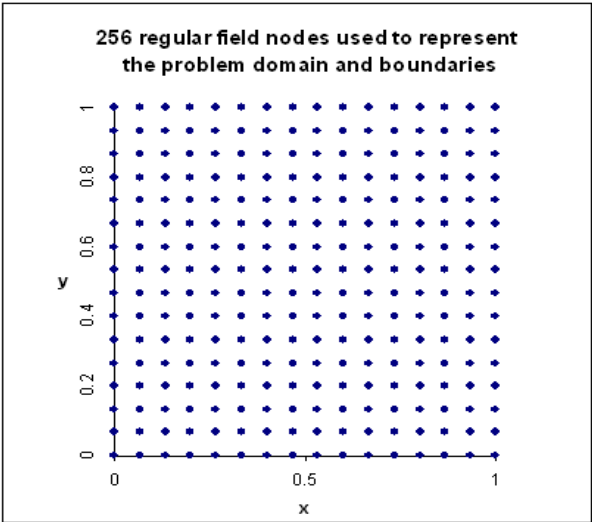


Fig. 3. The problem is represented by 256(16x16) regular nodes

5.1.3 Problem 1 with essential boundary conditions

Fig.4 shows the problem 1 and its boundary conditions. The temperature distribution in the plate obtained by MWS method presented in this chapter is given in Fig. 5. The constant temperature lines in this figure are shown by solid lines. Table.2 compares the results obtained by collocation method and LRPIM method with those obtained by the analytical method.

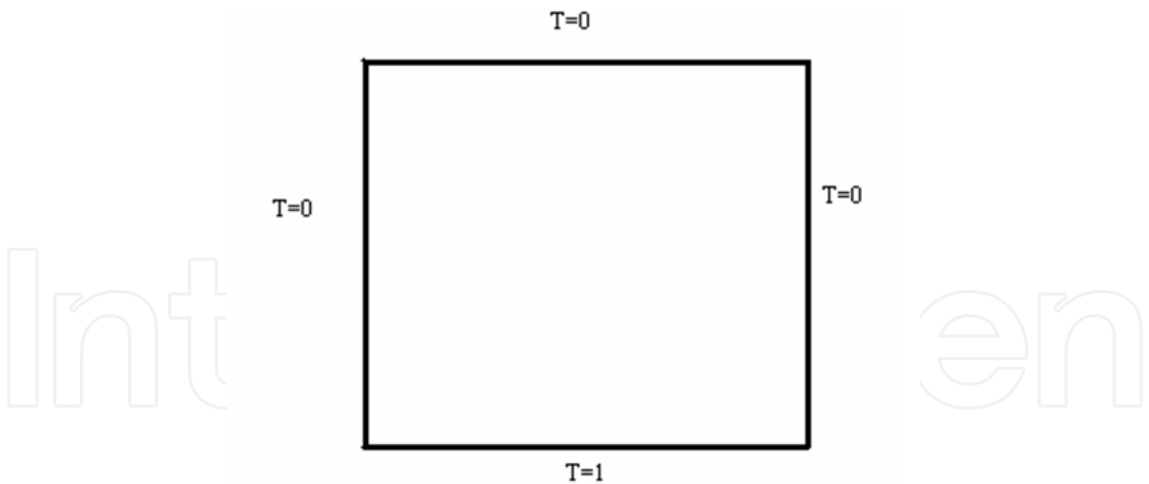


Fig. 4. Problem 1 and its boundary conditions

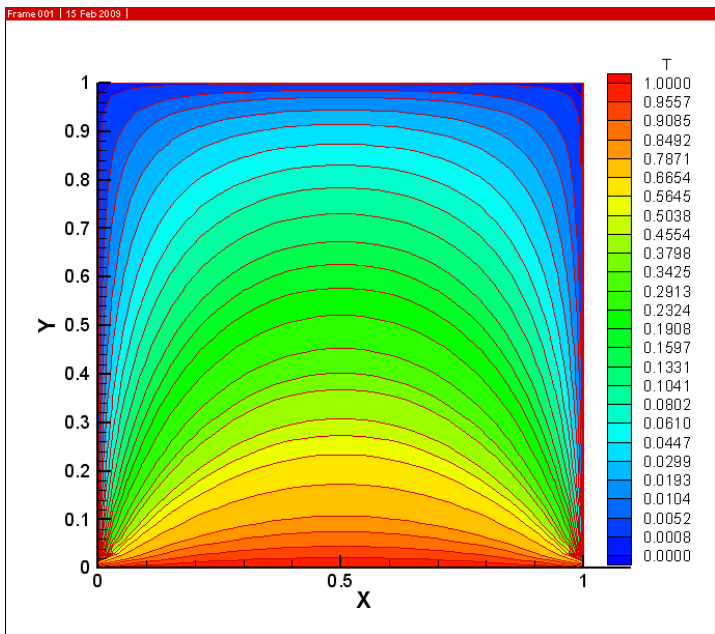


Fig. 5. Temperature distribution with essential boundary conditions (problem1 solved with MWS method)

the analytical solution of the problem can be written as

$$T(x,y)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-1^{n+1}+1}{n}\sin(n\pi x)(-\tanh(n\pi).\cosh(n\pi y)+\sinh(n\pi y)) \tag{24}$$

(x,y) are the coordinate of points in the plate. T is the temperature. We showed the difference between three meshfree methods and the difference between using different number of nodes to give the better results. We used the error norm

$$Err(j)=\frac{|T(j)-T_{analytic}(j)|}{T_{analytic}(j)} \tag{25}$$



and Total Error defined as

$$Total\ Err = \frac{(\sum_{j=1}^n Err(j))}{n}$$

(26)

x=0.5		256 nodes			1156 nodes	
y	Analytical	Collocation	LRPIM	MWS	LRPIM	MWS
0	1	1	1	1	1	1
0.1	0.8017	0.7978	0.7980	0.7976	0.8010	0.8013
0.2	0.6208	0.6137	0.6142	0.6134	0.6198	0.6203
0.3	0.4679	0.4624	0.4632	0.4620	0.4667	0.4671
0.4	0.3449	0.3390	0.3401	0.3386	0.3445	0.3448
0.5	0.2500	0.2462	0.2473	0.2458	0.2493	0.2495
0.6	0.1765	0.1728	0.1737	0.1724	0.1760	0.1761
0.7	0.1194	0.1174	0.1181	0.1171	0.1190	0.1191
0.8	0.0737	0.0720	0.0725	0.0717	0.0735	0.0736
0.9	0.0351	0.0345	0.0419	0.0342	0.0350	0.0350
1	0	0	0	0	0	0

Table 2. Comparison between Mfree methods and analytical method in problem 1

x=0.5		256 nodes			1156 nodes		
y	Analytical Err	Collocation Err	LRPIM Err	MWS Err	Collocation Err	LRPIM Err	MWS Err
0.1	0	0.0051	0.0046	0.0051	0.0015	0.0009	0.0004
0.2	0	0.0119	0.0106	0.0119	0.0029	0.0016	0.0008
0.3	0	0.0126	0.0100	0.0126	0.0045	0.0026	0.0009
0.4	0	0.0183	0.0139	0.0183	0.0038	0.0012	0.0009
0.5	0	0.0168	0.0108	0.0168	0.0056	0.0028	0.0008
0.6	0	0.0232	0.0159	0.0232	0.0062	0.0028	0.0006
0.7	0	0.0193	0.0109	0.0193	0.0067	0.0034	0.0008
0.8	0	0.0271	0.0163	0.0271	0.0068	0.0027	0.0014
0.9	0	0.0256	0.1937	0.0256	0.0085	0.0028	0.0001
Total Err		0.018	0.0319	0.0178	0.005	0.0023	0.0007

Table 3. Errors in problem 1

5.1.4 Problem 2 with natural boundary condition

The MWS method is used to solve the same problem with both essential and natural boundary conditions by 256(16x16) and 1156(34x34) nodes. The temperature distributions for those problems are given in Fig.7. It should be noted that the essential boundary conditions are satisfied exactly whereas the natural (Neumann) boundary conditions are satisfied in the weak form formulation.

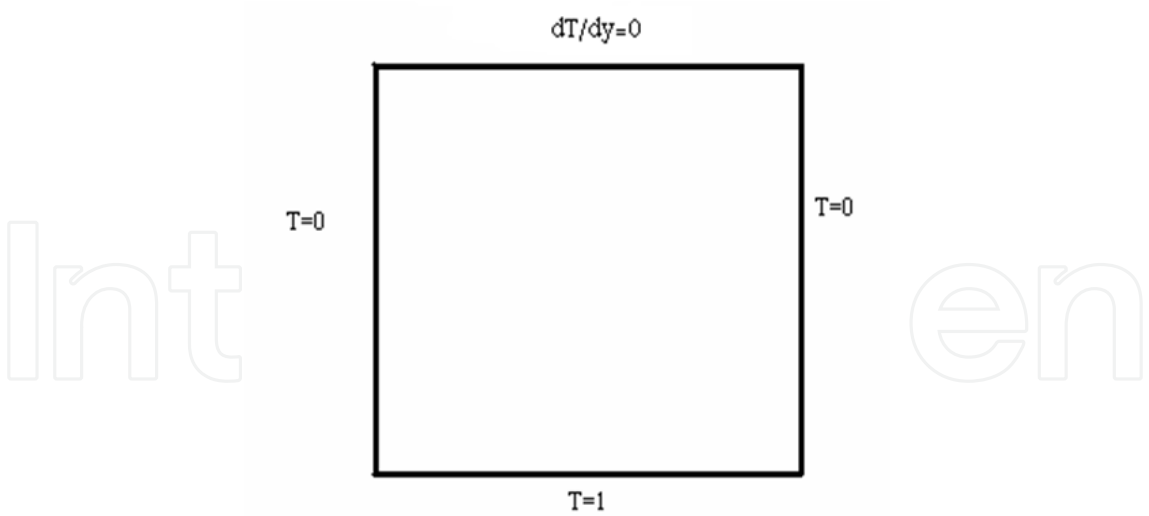


Fig. 6. Problem 2 and its boundary conditions

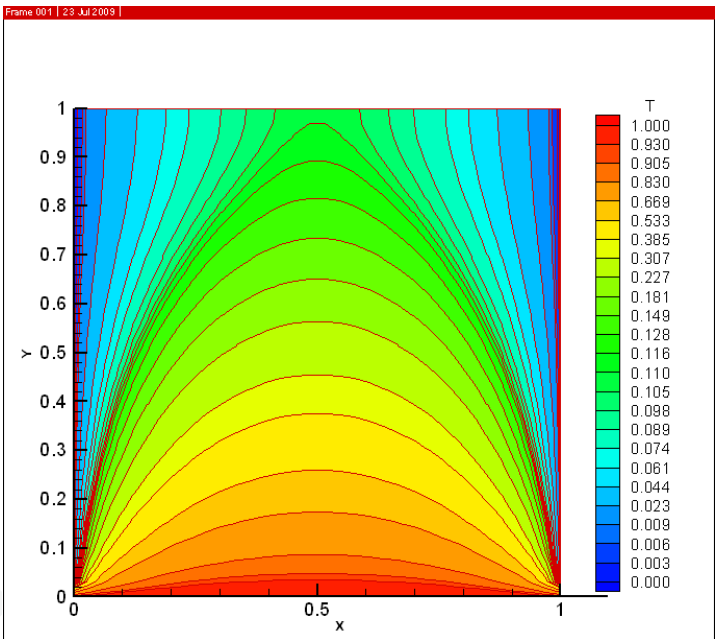


Fig. 7. Temperature distribution with 1156 nodes (Problem2 solved with MWS method)

In Tables 4 and 5 the LRPIM and MWS methods are compared with the analytical method. The numerical values for the temperature distributions with 256 and 1156 nodes are also given in Tables 4 and 5. The defined error equations (25 and 26) are used to show the accuracy of MWS and LRPIM.

5.2 Lid driven cavity problem

In this section, the lid driven cavity problem is solved by meshfree method. In this problem, the cavity is full of fluid and the upper plate in the cavity drive horizontally. It is shown that the moving boundary conditions in the top wall are easily applied and natural boundary condition can be satisfied.

x=0.5		256 nodes		1156 nodes	
y	analytical	MWS	LRPIM	MWS	LRPIM
0	1	1	1	1	1
0.1	0.8047	0.8006	0.8009	0.8035	0.8041
0.2	0.6271	0.6196	0.6202	0.6254	0.6262
0.3	0.4782	0.4723	0.4732	0.4762	0.4771
0.4	0.3606	0.3538	0.3550	0.3589	0.3598
0.5	0.2718	0.2675	0.2688	0.2705	0.2712
0.6	0.2071	0.2026	0.2039	0.2059	0.2066
0.7	0.1617	0.1591	0.1603	0.1608	0.1614
0.8	0.1320	0.1293	0.1304	0.1314	0.1318
0.9	0.1152	0.1137	0.1147	0.1147	0.1151
1	0.1098	0.1081	0.1089	0.1093	0.1097

Table 4. Comparison between Meshfree methods and analytical method in problem 2

x=0.5		256 nodes		1156 nodes	
y	Analytical Err	MWS Err	LRPIM Err	MWS Err	LRPIM Err
0.1	0	0.0051	0.0047	0.0015	0.0007
0.2	0	0.0120	0.0110	0.0027	0.0014
0.3	0	0.0123	0.0105	0.0042	0.0023
0.4	0	0.0189	0.0155	0.0047	0.0022
0.5	0	0.0158	0.0110	0.0048	0.0022
0.6	0	0.0217	0.0155	0.0058	0.0024
0.7	0	0.0161	0.0087	0.0056	0.0019
0.8	0	0.0205	0.0121	0.0045	0.0015
0.9	0	0.0130	0.0043	0.0043	0.0009
1	0	0.0158	0.0082	0.0046	0.0009
Total Err		0.0162	0.0108	0.0046	0.0017

Table 5. Errors in problem 2

5.2.1 Formulation and boundary conditions of driven cavity problem

The application of Navier-Stokes equation in solving fluid flow has evolved in the past few decades with meshfree method as one of the most adopted techniques. In this section the Navier-Stokes equation is solved:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial P}{\partial x} + 1/\text{Re}(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

(27)

The boundary conditions are shown in fig.8 and it is shown that three walls are without motion and the upper wall move with the fix speed.

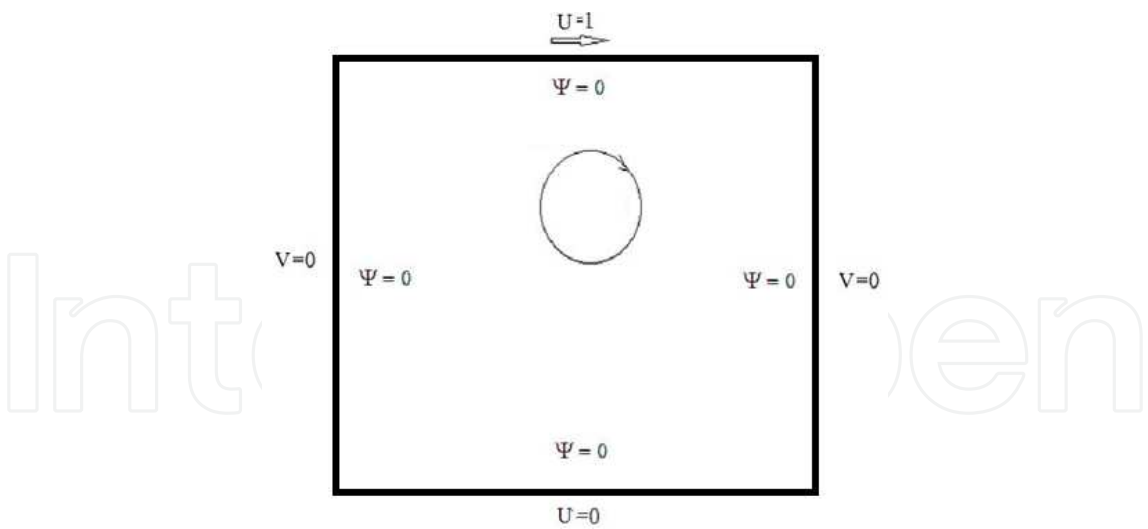


Fig. 8. Boundary conditions for driven cavity problem

5.2.2 Numerical results

The results from the solution of driven cavity problem when Reynolds number is 100 are shown below:

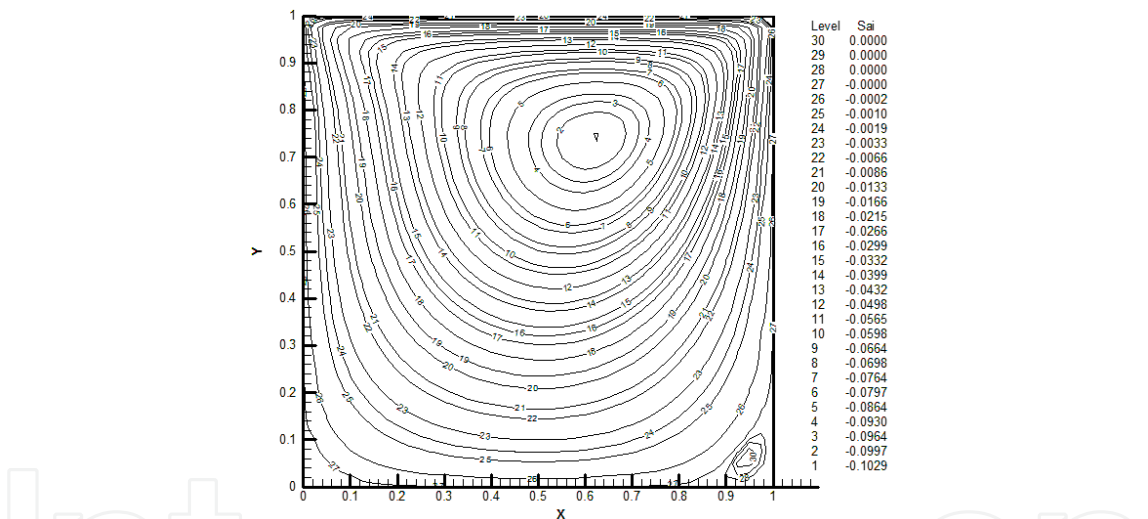


Fig. 9. Stream line contours for Reynolds 100

The vortex on the corner is created and is related to Reynolds number and the corner vortex will grow if Reynolds number increases. The vorticity and velocity contours are shown:

Location of large vortex	Minimum stream function	Reynolds	Reference
(0.6172,0.7344)	-0.0103	100	Ghia and Shin
(0.6196,0.7373)	-0.0103	100	Hou and Doolen
(0.6231,0.7460)	-0.0129	100	Present
(0.5547,0.6055)	-0.114	400	Ghia
(0.5500,0.6125)	-0.113	400	Gupta and Kalita
(0.5411,0.6005)	-0.114	400	Present

Table 6. Minimum stream function and location of large vortex

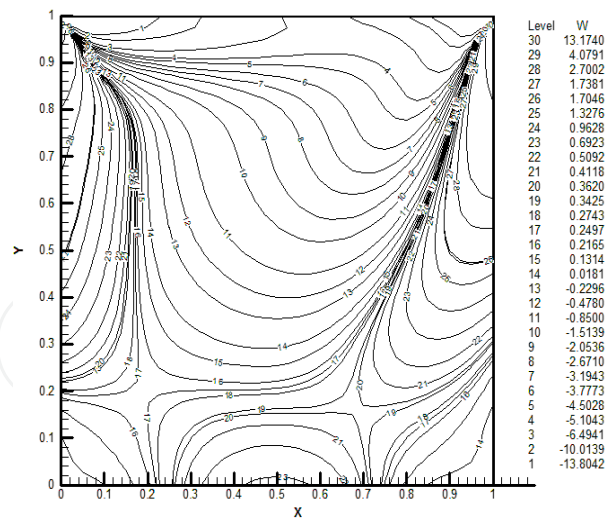


Fig. 10. Vorticity contours for driven cavity problem for Reynolds 100

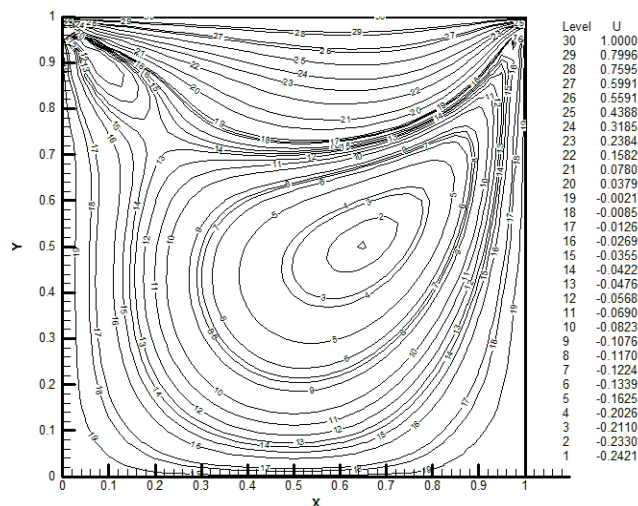


Fig. 11. Horizontal velocity contours for driven cavity problem for Reynolds 100

In Table 6 the results are compared with the minimum stream function and the location of large vortex. The results show the ability of meshfree methods to simulate the fluid mechanics problems.

5.3 Cantilever beam problem

Numerical studies are conducted for a cantilever beam that is often used for benchmarking numerical methods because the analytic solution for this problem is known. This problem is a sample of solid mechanics.

5.3.1 Formulation of cantilever beam problem

The equilibrium equation is used with the formula:

$$\sigma_{ij,j} + b_i = 0 \tag{28}$$

$\sigma$  is the stress vector and  $b_i$  is the body force vector components. The strain-displacement relations are another formula that are brought in two directions:



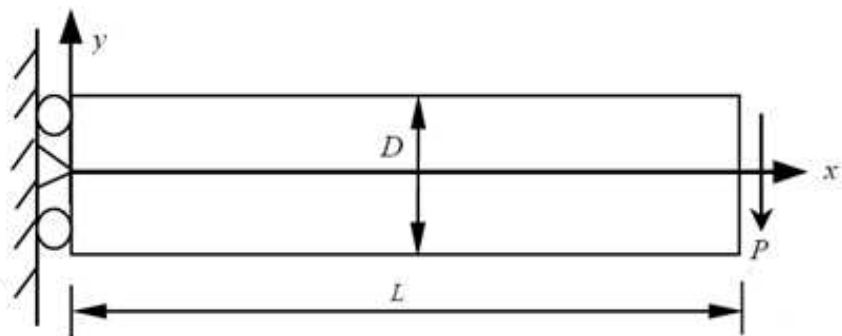


Fig. 12. The beam problem

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{29}$$

$\varepsilon_{xx}$  is the strain component and  $u$  is the displacement in the horizontal direction.

$$\varepsilon_{yy} = \frac{\partial v}{\partial x} \tag{30}$$

$\varepsilon_{yy}$  is the strain component and  $v$  is the displacement in the vertical direction.  
The last equation is Hook’s law:

$$\sigma = D_e \varepsilon \tag{31}$$

$D_e$  is the matrix of elastic constant.

5.3.2 Numerical results

The analytical solution is obtained for displacement of points of beam:

$$u = \frac{Py}{6EI} \left[ (6L - 3x)x + (2 + \nu)(y^2 - \frac{D^2}{4}) \right] \tag{32}$$

$u$  is the displacement of points in horizontal direction and  $P$  is the force at the end of the beam.  $E$  is the elasticity modulus and  $\nu$  is the poisson ratio and moment of inertia is  $I$ ,  $D$  is the height and  $L$  is the length

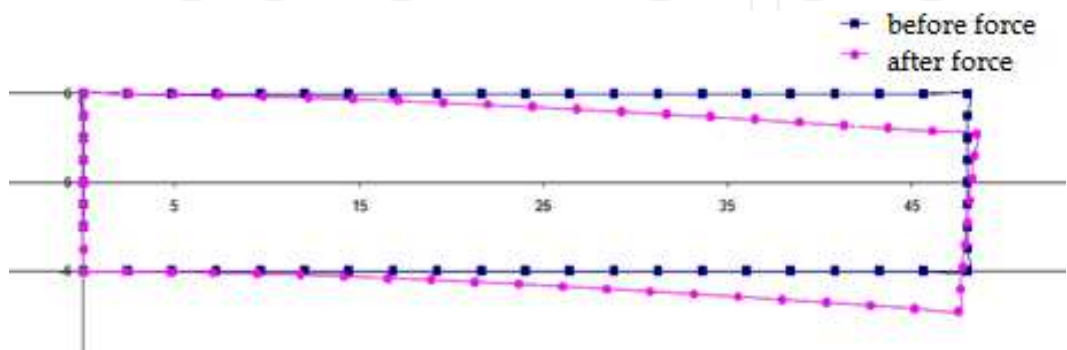


Fig. 13. The beam after effect of force

The energy norm is defined to compare the results.

$$e_n = \sqrt{\frac{1}{2} \left( \int_{\Omega} (\varepsilon_{numer} - \varepsilon_{exact}) D_e (\varepsilon_{numer} - \varepsilon_{exact}) d\Omega \right)}$$

(33)

$\Omega$  is the problem domain and  $\varepsilon_{num}$  and  $\varepsilon_{exact}$  are the strain vector with numerical and analytical solutions.

Energy norm	Solution method	No.
0.0258	MWS	1
0.026	LRPIM	2

Table 7. Energy norm for beam problem

These results show the capacity of MWS and LRPIM methods to solve the solid mechanics problems. The Table 7 shows the errors are minimums with comparison with the analytic solution.

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7. Conclusion

The meshfree methods are numerical methods that can be used to solve the many different and complicated problems. The heat transfer problems, solid and fluid mechanics problems have been solved with meshfree methods.

Three categories are used to solve the problems. Strong form methods, weak form methods and weak-strong form methods(MWS) are meshfree categories. They can be used to solve the problems with Dirichlet and Neumann boundary conditions. For examples the heat conduction problem and lid driven cavity and cantilever beam are solved that they have different type of boundary conditions. Solutions are related to many parameters: the selected meshfree method, number of nodes, shape function parameters et cetera.

Nowadays, many changes are employed to different types of meshfree methods. The advantages are improved and the high convergence rate and high accuracy are accessible.

8. References

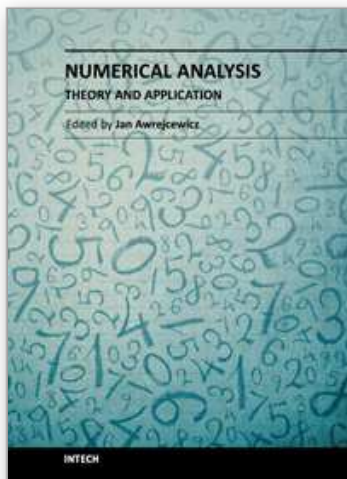
Hou, S. , Doolen, G. & Cogley, A. (1995). Simulation of cavity flows by lattice boltzmann method, *Journal of computational physics*, Vol.118, pp. 329-347

Gu, Y.T. & Liu, G.R. (2005). A meshfree weak-strong (MWS) form method for time dependent problems, *Computational Mechanics*, Vol. 35, No.2 , pp. 134-145

Gu, Y.T & Liu, G.R. (2001). A Local Point Interpolation Method (LPIM) For Static And Dynamic Analysis Of Thin Beams, *Computer Methods in Applied Mechanics and Engineering*. Vol. 190

- Hong, W.u. & Quan, W. (2007) Meshless method based on local weak forms for steady-state heat conduction problems, *International Journal of Heat and Mass Transfer* Vol.51, (2008) pp. 3103-3112
- Incropera, Frank. & Witt, David.P. (2002). *Introduction to heat transfer*, 4<sup>th</sup> edition, springer
- Liu, G.R. & Gu, Y.T. (2005). *An Introduction to Meshfree Methods and Their Programming*, springer
- Liu, G.R. , Wu, Y.L. & Ding, H. (2005). Meshfree weak-strong(MWS) form method and its application to incompressible flow problems, *International Journal for Numerical Methods in Fluids*, Vol.46, pp. 1025-1047
- Liu, G.R. & Gu, Y.T. (2003). A meshfree method: Meshfree Weak-Strong (MWS) form method, for 2-D solids, *Computational Mechanics*, Vol.33, No.1, pp. 2-14
- Liu, G.R. , Yan, L. , Wang, J.G. & Gu, Y.T. (2002). Point Interpolation Method Based On Local Residual Formulation Using Radial Basis Functions, *Structure Engineering Mechanic*, Vol.14, No.6, pp. 713-732
- Liu, G.R. & Gu, Y.T. (2001). A local radial point interpolation method (LRPIM) for free vibration analyses of 2-D solids, *Journal of Sound and Vibration*, Vol.246, No.1, pp. 29-46
- Liu, G.R. & Gu, Y.T. (2003). A meshfree formulation of local radial point interpolation method (LRPIM) for incompressible flow simulation, *Computational Mechanics* Vol.30, pp. 355-365
- Rao, S. (2004), *The Finite Element Method in Engineering*, Elsevier Science & Technology Books.
- Reddy, J.N. (2006), *An introduction to finite element method*, Third edition, Springer, McGraw-Hill Publishing Corporation
- Zahiri, S. , Daneshmand, F. and Akbari, M.H. (2009). Using meshfree weak-strong form method for 2-D heat transfer problem, *Proceedings of ASME 2009 International Mechanical Engineering Congress and Exposition*, IMECE2009-12525, Lake Buena Vista, Florida, USA, November 13-19, 2009

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